

Scale-dependent metric and minimal length in QEG

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2007 J. Phys. A: Math. Theor. 40 6595

(<http://iopscience.iop.org/1751-8121/40/25/S04>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.109

The article was downloaded on 03/06/2010 at 05:15

Please note that [terms and conditions apply](#).

Scale-dependent metric and minimal length in QEG

Martin Reuter and Jan-Markus Schwindt

Institute of Physics, University of Mainz, D-55128 Mainz, Germany

Received 5 October 2006, in final form 16 November 2006

Published 6 June 2007

Online at stacks.iop.org/JPhysA/40/6595

Abstract

The possibility of a minimal physical length in quantum gravity is discussed within the asymptotic safety approach. Using a specific mathematical model for length measurements ('COM microscope'), it is shown that the spacetimes of quantum Einstein gravity based upon a special class of renormalization group trajectories are 'fuzzy' in the sense that there is a minimal coordinate separation below which two points cannot be resolved.

PACS numbers: 04.60.-m, 11.10.Hi, 98.80.Cq

1. Introduction

It is an old speculation [1] that quantum gravity induces a lower bound on physically realized distances. Since this issue can be addressed only in a fundamental quantum theory of gravity (as opposed to a low energy effective theory), it is natural to analyse it within quantum Einstein gravity (QEG). This theory is an attempt at the nonperturbative construction of a predictive quantum field theory of the metric by means of a non-Gaussian renormalization group (RG) fixed point [2–18]. From what is known today, it appears indeed increasingly likely that there does exist an appropriate fixed point which makes QEG nonperturbatively renormalizable or 'asymptotically safe' [5, 7, 8, 10]. The asymptotic safety scenario for QEG is most conveniently formulated in the language of Wilson's general framework of renormalization [19], using an 'exact renormalization group equation' which defines an RG flow on the infinite dimensional 'theory space' consisting of all action functionals satisfying certain symmetry constraints. The key idea is to base the construction of the theory on a trajectory running inside the unstable manifold (ultraviolet critical hypersurface) of a non-Gaussian fixed point of the RG flow. In the extreme ultraviolet (for the RG scale $k \rightarrow \infty$) it starts infinitely close to the fixed point, and by successive coarse graining steps it is driven away from it, thus lowering the scale k . Conversely, starting at some finite k and increasing the energy or momentum scale, the trajectory gets attracted to the fixed point. As a result of this 'benign' high energy, i.e. short distance behaviour, the theory is asymptotically (i.e. for $k \rightarrow \infty$) safe from unphysical divergences [20].

An important tool in analysing the RG flow of QEG is the effective average action and its exact functional RG equation [21, 22]. In the case of QEG [3], the average action is

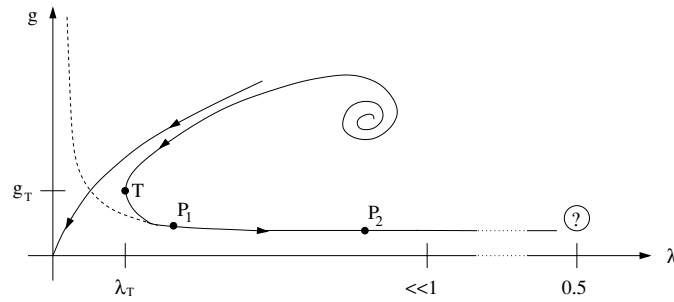


Figure 1. A Type IIIa trajectory and the separatrix. The dashed line is a trajectory of the canonical RG flow. The arrows point in the direction of decreasing k .

a diffeomorphism invariant functional of the metric, $\Gamma_k[g_{\mu\nu}]$, which depends on a variable infrared (IR) cutoff k . For $k \rightarrow \infty$ it approaches the bare action S , while it equals the standard effective action at $k = 0$. At least in Euclidean non-gauge theories on flat space, Γ_k at intermediate scales has the following properties [23]. (i) It defines an effective field theory at the momentum scale k . This means that every physical process which involves only a single momentum scale, say p , is well described by a tree level evaluation of Γ_k with k chosen as $k = p$. (ii) At least heuristically [23], Γ_k may be interpreted as arising from a continuum version of a Kadanoff–Wilson block spin procedure, i.e. it defines the dynamics of ‘coarse grained’ dynamical variables which are averaged over a certain region of Euclidean spacetime. Denoting the typical linear extension of the averaging region by ℓ , one has $\ell \approx \pi/k$ in flat spacetime. In this sense, Γ_k can be thought of as a ‘microscope’ with an adjustable resolving power $\ell = \ell(k)$.

In quantum gravity where the metric is dynamical, the relationship between the IR cutoff k and the ‘averaging scale’ ℓ is more complicated in general. In the following, we shall review a concrete definition of an ‘averaging’ or ‘coarse graining’ proper length scale $\ell = \ell(k)$. Using this definition, along with certain qualitative properties of the RG trajectories of QEG, we shall demonstrate that the theory generates a minimal length scale in a dynamical way. The interpretation of this scale is rather subtle, however. One has to carefully distinguish different physical questions one could ask, because depending on the question a minimal length will or will not become visible. Our presentation follows [24].

The running action $\Gamma_k[g_{\mu\nu}]$ can be obtained from an exact functional RG equation [3]. In practice it is usually solved on a truncated theory space. In the Einstein–Hilbert truncation, for instance, Γ_k is approximated by a functional of the form

$$\Gamma_k[g, \bar{g}] = (16\pi G(k))^{-1} \int d^4x \sqrt{\bar{g}} \{-R(g) + 2\Lambda(k)\} + S_{\text{gf}}[g, \bar{g}], \quad (1)$$

involving a running Newton constant $G(k)$ and cosmological constant $\Lambda(k)$. Here S_{gf} is a background gauge fixing term, depending on the background metric $\bar{g}_{\mu\nu}$, whose form is kept classical for all k . It contains a constant gauge fixing parameter α . Likewise the truncation ansatz contains the classical Faddeev–Popov ghost action corresponding to the background gauge (not written out in equation (1)).

The qualitative properties of the trajectories following from the Einstein–Hilbert approximation are well known by now [6]. Figure 1 shows a ‘Type IIIa’ trajectory which would be the type that is presumably realized in the real universe since it is the only type that has a positive Newton’s constant $G(k)$ and a small positive cosmological constant $\Lambda(k)$ at macroscopic scales. In figure 1 it is plotted in terms of the dimensionless parameters

$g(k) \equiv k^2 G(k)$ and $\lambda(k) \equiv \Lambda(k)/k^2$ and compared to the canonical trajectory (dashed curve) with $\Lambda = \text{const}$ and $G = \text{const}$. The Type IIIa trajectory contains the following four parts, with increasing values of the cutoff k .

- (i) The classical regime for small k where the trajectory is identical to the canonical one (in figure 1, the segment between the points P_1 and P_2).
- (ii) The turnover regime where the trajectory, close to the Gaussian fixed point at $g = \lambda = 0$, begins to depart from the canonical one and turns over to the ‘separatrix’ which connects the Gaussian with the non-Gaussian fixed point (g_*, λ_*) . By definition, the coordinates of the turning point T are g_T and λ_T , and it is passed at the scale $k = k_T$.
- (iii) The growing Λ regime where $G(k)$ is approximately constant, but $\Lambda(k)$ runs proportional to k^4 .
- (iv) The fixed point regime where the trajectory approaches the non-Gaussian fixed point in an oscillating manner. Directly at the fixed point one has $g(k) \equiv g_*$ and $\lambda(k) \equiv \lambda_*$, and therefore $G(k) \propto k^{-2}$ and $\Lambda(k) \propto k^2$ for $k \rightarrow \infty$. The non-Gaussian fixed point is responsible for the nonperturbative renormalizability of the theory.

The behaviour of the trajectory in the extreme infrared is not yet known since the Einstein–Hilbert approximation breaks down when $\lambda(k)$ approaches the value $1/2$. A more general truncation is needed to approximate the RG trajectory in that region. For this reason the classical region (i) does not necessarily extend to $k = 0$, and we speak about ‘laboratory’ scales for values of $k \equiv k_{\text{lab}}$ in the region where G and Λ are constant. The Planck mass is then defined as $m_{\text{Pl}} \equiv 1/\sqrt{G(k_{\text{lab}})}$.

In the regimes (i), (ii) and (iii) the trajectory is well approximated by linearizing the RG flow about the Gaussian fixed point. In terms of the dimensionful parameters, one finds that in its linear regime $G(k) = \text{const}$ and [25]

$$\Lambda(k) = \Lambda_0[1 + (k/k_T)^4], \quad (2)$$

where Λ_0 is a constant. The corresponding dimensionless $\lambda = \Lambda/k^2$ runs according to

$$\lambda(k) = \Lambda_0[(1/k)^2 + (k/k_T^2)^2]. \quad (3)$$

Note that this function is invariant under the ‘duality transformation’ $k \mapsto k_T^2/k$:

$$\lambda(k) = \lambda(k_T^2/k). \quad (4)$$

For further details and a discussion of the other types of trajectories, see [6, 25]. The analysis in the following sections refers entirely to trajectories of Type IIIa.

Let us briefly comment on more general truncations. Above we illustrated the general ideas and constructions underlying truncated gravitational RG flows by means of the simplest example, the Einstein–Hilbert truncation (1). The flow equations for the higher derivative R^2 truncation are likewise known in a closed form [7, 8], and their solutions reveal important evidence for asymptotic safety. They confirm the picture drawn above.

Concerning generalizations of the ghost sector truncation, no results are available yet but there is a partial result concerning the gauge fixing term. Even if one makes the ansatz (1) for $\Gamma_k[g, \bar{g}]$ in which the gauge fixing term has the classical (or more appropriately, bare) structure $S_{\text{gf}} = \frac{1}{2\alpha} \int d^4x \sqrt{\bar{g}} \bar{g}^{\mu\nu} Q_\mu Q_\nu$ with any background-type gauge condition Q_μ , one should treat its prefactor as a running coupling: $\alpha \equiv \alpha_k$. The beta function of α has not been determined yet from the exact RG equation for Γ_k , but there is a simple argument which allows us to bypass this calculation.

In nonperturbative Yang–Mills theory and in perturbative quantum gravity, $\alpha = \alpha_* = 0$ is known to be a fixed point for the α evolution. The following heuristic argument suggests that the same should be true beyond perturbation theory for the functional integral defining

the effective average action for gravity. In the standard functional integral the limit $\alpha \rightarrow 0$ corresponds to a sharp implementation of the gauge fixing condition, i.e. $\exp(-S_{\text{gf}})$ becomes proportional to $\delta[Q_\mu]$. The domain of the $\mathcal{D}h_{\mu\nu}$ integration over metric fluctuations $h_{\mu\nu}$ consists of those $h_{\mu\nu}$'s which satisfy the gauge fixing condition exactly, $Q_\mu = 0$. Adding the infrared cutoff at k amounts to suppressing some of the $h_{\mu\nu}$ modes while retaining the others. But since all of them satisfy $Q_\mu = 0$, a variation of k cannot change the domain of the $h_{\mu\nu}$ integration. The delta functional $\delta[Q_\mu]$ continues to be present for any value of k if it was there originally. As a consequence, α vanishes for all k , i.e. $\alpha = 0$ is a fixed point of the α evolution [26]. In other words, we can mimic the dynamical treatment of a running α by setting the gauge fixing parameter to the constant value $\alpha = 0$. The calculation for $\alpha = 0$ is more complicated than at $\alpha = 1$, but for the Einstein–Hilbert truncation the α -dependence of β_g and β_λ for arbitrary constant α has been found both for ‘type A’ and ‘type B’ cutoffs [5]. The R^2 truncation could be analysed only in the simpler $\alpha = 1$ gauge, but the results from the Einstein–Hilbert truncation suggest that the quantities of interest do not change much between $\alpha = 0$ and $\alpha = 1$ [7, 8]. Only when one goes far away from the ‘physical’ point $\alpha = 0$, the Einstein–Hilbert truncation shows a significant α -dependence which can even lead to qualitative changes in the properties of the beta functions [27].

2. Mean field metric and scale-dependent distances

Let us pick a specific RG trajectory, a curve $k \mapsto \Gamma_k$ on theory space. The effective field equations implied by Γ_k define a k -dependent expectation value of the metric, a kind of mean field, $\langle g_{\mu\nu} \rangle_k$:

$$\frac{\delta\Gamma_k}{\delta g_{\mu\nu}(x)}[\langle g \rangle_k] = 0. \quad (5)$$

In the Einstein–Hilbert truncation (1), these equations are

$$R_{\mu\nu}(\langle g \rangle_k) = \Lambda(k)\langle g_{\mu\nu} \rangle_k. \quad (6)$$

The infinitely many equations in (5), one at each scale k , are valid simultaneously, and all the mean fields $\langle g_{\mu\nu} \rangle_k$ refer to one and the same physical system, a ‘quantum spacetime’ in the QEG sense. The mean fields $\langle g_{\mu\nu} \rangle_k$ describe the metric structure in dependence on the length scale on which the spacetime manifold is probed. An observer exploring the structure of spacetime using a ‘microscope’ of resolution $\ell(k)$ will perceive the universe as a Riemannian manifold with the metric $\langle g_{\mu\nu} \rangle_k$. While $\langle g_{\mu\nu} \rangle_k$ is a smooth classical metric at every fixed k , the quantum spacetime can have fractal properties because on different scales different metrics apply. In this sense, the metric structure on the quantum spacetime is given by an infinite set $\{\langle g_{\mu\nu} \rangle_k; 0 \leq k < \infty\}$ of ordinary metrics.

Recently it has been shown [28] that in asymptotically safe theories of gravity, at sub-Planckian distances, spacetime is indeed a fractal whose spectral dimension [29] equals 2. It is quite remarkable that a similar dynamical dimensional reduction from 4 macroscopic to 2 microscopic dimensions has also been observed in Monte Carlo simulations of causal dynamical triangulations [30–32]. (See also [33].)

In order to understand the relation between ℓ and the IR cutoff k , we must recall the essential steps in the construction of the average action [3]. The formal starting point is the path integral $\int \mathcal{D}\gamma_{\mu\nu} \exp(-S[\gamma])$ over all metrics $\gamma_{\mu\nu}$: gauge fixed by means of a background gauge fixing condition. Even without an IR cutoff, upon introducing sources and performing the usual Legendre transform one is led to an effective action $\Gamma[g_{\mu\nu}; \bar{g}_{\mu\nu}]$ which depends on two metrics, the expectation value of $\gamma_{\mu\nu}$, denoted $g_{\mu\nu}$, and the non-dynamical background

field $\bar{g}_{\mu\nu}$. The functional $\Gamma[g_{\mu\nu}] \equiv \Gamma[g_{\mu\nu}; \bar{g}_{\mu\nu} = g_{\mu\nu}]$ obtained by equating the two metrics generates a set of 1PI Green's functions for the theory.

The IR cutoff is implemented by first expanding the shifted integration variable $h_{\mu\nu} \equiv \gamma_{\mu\nu} - \bar{g}_{\mu\nu}$ in terms of eigenmodes of \bar{D}^2 , the covariant Laplacian formed with the background metric $\bar{g}_{\mu\nu}$, and interpreting $\mathcal{D}h_{\mu\nu}$ as an integration over all expansion coefficients. Then, a suppression term is introduced which damps the contribution of all \bar{D}^2 -modes with eigenvalues smaller than k^2 . Following the usual steps [3, 23] this leads to the scale-dependent functional $\Gamma_k[g_{\mu\nu}; \bar{g}_{\mu\nu}]$, and again the action with one argument is obtained by equating the two metrics: $\Gamma_k[g_{\mu\nu}] \equiv \Gamma_k[g_{\mu\nu}; \bar{g}_{\mu\nu} = g_{\mu\nu}]$. It is this action which appears in (5). Because of the identification of the two metrics it is, in a sense, the eigenmodes of D^2 , constructed from the argument of $\Gamma_k[g]$, which are cut off at k^2 . Note however that neither the $g_{\mu\nu}$ - nor the $\bar{g}_{\mu\nu}$ -argument of Γ_k has any dependence on k . Therefore, $\gamma_{\mu\nu}$ is expanded in terms of the eigenfunctions of a *fixed* operator \bar{D}^2 . Since its eigenfunctions are complete, we really integrate over all metrics when we lower k from infinity to zero. Also note that a k -dependent mean field arises only at the point where we go 'on shell' with $g_{\mu\nu} = \bar{g}_{\mu\nu}$: the solution $\langle g_{\mu\nu} \rangle_k$ to equation (5) depends on k , simply because Γ_k does so.

In [28] an algorithm was proposed which allows the reconstruction of the 'averaging' scale ℓ from the cutoff k . The input data are the set of metrics characterizing a quantum manifold $\{\langle g_{\mu\nu} \rangle_k\}$. The idea is to deduce the relation $\ell = \ell(k)$ from the spectral properties of the *scale-dependent* Laplacian $\Delta_k \equiv D^2(\langle g_{\mu\nu} \rangle_k)$ built with the solution of the effective field equation. More precisely, for every fixed value of k , one solves the eigenvalue problem of $-\Delta_k$ and studies in particular the properties of the eigenfunctions whose eigenvalue is k^2 , or nearest to k^2 in the case of a discrete spectrum. We shall refer to an eigenmode of $-\Delta_k$ whose eigenvalue is (approximately) the square of the cutoff k as a 'cutoff mode' (COM) and denote the set of all COMs by $\text{COM}(k)$.

If we ignore the k -dependence of Δ_k for a moment (as would be appropriate for matter theories in flat space) the COMs are, for a sharp cutoff, precisely the last modes integrated out when lowering the cutoff, since the suppression term in the path integral cuts out all $h_{\mu\nu}$ -modes with an eigenvalue smaller than k^2 .

For a non-gauge theory in flat space the coarse graining or averaging of fields is a well-defined procedure, based upon ordinary Fourier analysis, and one finds that in this case the length ℓ is essentially the wavelength of the last modes integrated out, the COMs.

This observation motivates the following tentative *definition* of ℓ in quantum gravity. We determine the COMs of $-\Delta_k$, analyse how fast these eigenfunctions vary on spacetime and read off a typical coordinate distance Δx^μ characterizing the scale on which they vary. For an oscillatory COM, for example, Δx would correspond to an oscillation period. Finally, we use the metric $\langle g_{\mu\nu} \rangle_k$ itself in order to convert Δx^μ to a proper length. This proper length, by definition, is $\ell(k)$. The experience with theories in flat spacetime suggests that the COM scale ℓ is a plausible *candidate* for a physically sensible resolution function $\ell = \ell(k)$, but there might also be others, depending on the experimental setup one has in mind.

In a quantum spacetime, the (geodesic, say) distance of two given points x and y depends on k :

$$L_k(x, y) \equiv \int_{\mathcal{C}_{xy}^{(k)}} (\langle g_{\mu\nu} \rangle_k dx^\mu dx^\nu)^{1/2}. \tag{7}$$

Here, $\mathcal{C}_{xy}^{(k)}$ denotes the (possibly k -dependent) geodesic connecting x to y . The interpretation of this k -dependent distance is as follows. If k parametrizes the 'resolution of the microscope' with which the spacetime is observed, the metric $\langle g_{\mu\nu} \rangle_k$ and correspondingly the distance $L_k(x, y)$ pertain to a specific scale of resolution, and different observers, using microscopes

of different k -values, will measure different lengths in general. This k -dependence of lengths is analogous to the ‘coastline of England phenomenon’ well known from fractal geometry [34, 29].

3. A minimal length on the QEG 4-sphere

The QEG 4-sphere [24] is a manifold in the QEG sense, i.e. supplied with a family of infinitely many metrics $\{\langle g_{\mu\nu} \rangle_k | k = 0, \dots, \infty\}$. To be specific, it is the family of maximally symmetric solutions of (6) with positive curvature. It only exists provided $\Lambda(k) > 0$, which is the case for all Type IIIa trajectories.

We may parametrize S^4 by coordinates $(\zeta, \eta, \theta, \phi)$ with ranges $0 < \zeta, \eta, \theta < \pi$ and $0 \leq \phi < 2\pi$. The line element $\langle ds^2 \rangle_k \equiv \langle g_{\mu\nu} \rangle_k dx^\mu dx^\nu$ can be written as

$$\langle ds^2 \rangle_k = r^2(k)[d\zeta^2 + \sin^2 \zeta (d\eta^2 + \sin^2 \eta (d\theta^2 + \sin^2 \theta d\phi^2))], \quad (8)$$

where $r(k)$ is the k -dependent radius of S^4 implied by (6)

$$r(k) = \sqrt{3/\Lambda(k)}. \quad (9)$$

The family of metrics (8), (9) constitutes a concrete example of a quantum spacetime, as was discussed in [28]. In contrast to a Brownian curve or the coastline of England, distances *decrease* when we *increase* the cutoff k . The metric scales as $\langle g_{\mu\nu} \rangle_k \propto 1/\Lambda(k)$ so that in the fixed point regime $\langle g_{\mu\nu} \rangle_k \propto 1/k^2$, implying $L_k(x, y) \propto 1/k$ for any (geodesic) distance. On the equator, $\zeta = \eta = \theta = \pi/2$, the geodesic distance (7) of two points x and y with angles $\phi(x)$ and $\phi(y)$ reads

$$L_k(x, y) = \sqrt{3/\Lambda(k)}|\phi(x) - \phi(y)| = k^{-1}\sqrt{3/\lambda(k)}|\phi(x) - \phi(y)|. \quad (10)$$

On the quantum S^4 , the scalar eigenfunctions of $-\Delta_k$ are the spherical harmonics $Y_{n_1 l_2 m}(\zeta, \eta, \theta, \phi)$, labelled by four integer quantum numbers n, l_1, l_2 and m , where $n \geq l_1 \geq l_2 \geq |m|$. They have the eigenvalues

$$\mathcal{E}_n = n(n+3)/r^2(k), \quad n = 0, 1, 2, 3, \dots \quad (11)$$

The eigenvalues for the vector and tensor modes are slightly different, but for large n the difference becomes negligible and the spectrum is to a good approximation continuous. We will use this continuum approximation since we are interested in small angular distances $\Delta\phi$ anyway. Let us determine the associated set of cutoff modes $\text{COM}(k)$, i.e. the eigenfunctions with $-\Delta_k$ -eigenvalue as close as possible to k^2 . Inserting $\mathcal{E} \approx k^2$ into (11) and using equation (9) for $r(k)$, we find the following equation for the n -quantum number of the COMs at scale k :

$$n(k) \approx \sqrt{3/\Lambda(k)}k = \sqrt{3/\lambda(k)}. \quad (12)$$

Obviously $n(k)$ is indeed large if $\lambda(k) \ll 1$. The set $\text{COM}(k)$ consists of all harmonics $Y_{n_1 l_2 m}$ with n fixed by equation (12) and l_1, l_2 and m arbitrary.

Apart from its obvious dependence on the scale, the set $\text{COM}(k)$ depends on the RG trajectory via the function $\lambda(k)$ which determines $n(k)$. The function $\lambda = \lambda(k)$ is not invertible in general and different k 's can lead to the same $\text{COM}(k)$. Let us look at the Type IIIa trajectory in figure 1 as an example. First we concentrate on its part close to the turning point, staying away from the spiralling regime in the UV, and the IR region where the Einstein–Hilbert truncation breaks down. We then observe that for every scale $k < k_T$ below the turning point, there exists a corresponding scale $k^\sharp > k_T$ which has the same λ - and therefore n -value. As a result, the corresponding cutoff modes are equal at the two scales: $\text{COM}(k) = \text{COM}(k^\sharp)$. If the turning point is sufficiently close to the Gaussian fixed point, and k is not too far from

k_T , we may use the linearization (3) for an approximate determination of k^\ddagger . Because of the ‘duality symmetry’ (4), it is given by $k^\ddagger = k_T^2/k$. In the ‘spiralling’ regime, many different k -values have the same $\lambda(k)$ and COM(k).

Next, we determine the degree of position dependence of the COMs and quantify their ‘resolving power’. In order to convert the estimate for $n(k)$, equation (12), to an angular resolution, we note that it is sufficient to do so for one position and one direction. By the translation and rotation symmetries of the sphere, the resolution will be the same at any other point and in any other direction. We therefore choose to determine the angular resolution of the modes along the equator.

Two of the Δ_k -eigenfunctions with eigenvalue $n(k)$, namely $Y_\pm \equiv Y_{nn\pm n}$, oscillate most rapidly as a function of ϕ , and we shall use them in order to define the angular resolution. Their ϕ -dependence is $e^{\pm in\phi}$ and the corresponding angular resolution is

$$\Delta\phi(k) = \pi/n(k) = \pi\sqrt{\lambda(k)/3}. \quad (13)$$

As expected, the angular resolution implied by the COMs depends on the RG trajectory. It does so only via the function $\lambda = \lambda(k)$ and, as a result, can be of the same size for different values of k .

By definition, the COM scale ℓ is the *proper* length corresponding to $\Delta\phi(k)$ as computed with the metric $\langle g_{\mu\nu} \rangle_k$ of equations (8) and (9). From equations (10) and (13), we obtain

$$\ell(k) = \pi/k. \quad (14)$$

So we find that, as in theories on a classical flat spacetime, the natural proper length scale ℓ of the COM(k)-modes is just π/k . Thanks to the symmetry of the sphere, it is neither position nor direction dependent.

Taking the result $\ell \propto 1/k$, it seems as if nothing remarkable had happened. But the surprising effects appear in our result for the angular resolution, equation (13). As we can see from the flow diagram of figure 1, $\lambda(k)$ takes on a minimum value λ_T at the turning point T . In fact, as $\lambda(k) \geq \lambda_T$ for any scale k , we conclude that the angular resolution $\Delta\phi(k)$ is bounded below by the minimum angular resolution

$$\Delta\phi_{\min} = \pi\sqrt{\lambda_T/3}. \quad (15)$$

Stated differently, there does not exist any cutoff k for which $\Delta\phi(k)$ would be smaller than $\Delta\phi_{\min}$. On the other hand, angular resolutions between $\Delta\phi_{\min}$ and $\Delta\phi_* \equiv \pi\sqrt{\lambda_*/3}$ are realized for at least two scales k .

What has happened here? Coming from small k , we travel along the RG trajectory and follow its S^4 solutions, observing spacetime with a ‘microscope’ of variable proper resolution $\ell(k)$. At first, in the classical regime, an increase of k leads to the resolution of finer and finer structures since $\Lambda = \text{const}$ implies $\Delta\phi(k) \propto 1/k$. For the canonical RG trajectory, this behaviour would continue even for $k \rightarrow \infty$. In quantum gravity, however, in region (ii), the sphere starts to *shrink*, due to a growing cosmological constant $\Lambda(k)$. At the turning point scale k_T at which $\lambda(k)$ assumes its minimum λ_T , the shrinking becomes faster than the improvement of the resolution ($r(k) \propto k^{-2}$ in region (iii)). Although we can resolve smaller and smaller proper distances, this is of no use, since the sphere is shrinking so fast that a *smaller* proper length corresponds to a *larger* angular distance. Finally, in the fixed point regime (at large angles although this is an ultraviolet fixed point), the shrinking slows down to a rate that exactly cancels the improved resolution of the microscope ($r(k) \propto k^{-1}$) so that the angular resolution approaches a constant value $\Delta\phi_*$ after the oscillations have been damped away.

The minimum of $\Delta\phi$ at the turning point is equivalent to a maximum of the n quantum number the COMs can have: $n_{\max} \approx \sqrt{3/\lambda_T}$. This result does *not* mean that in the fundamental

path integral underlying the flow equation not all quantum fluctuations are integrated out when k is lowered from infinity to $k = 0$. It should instead be thought of as reflecting properties of the mean field $\langle g_{\mu\nu} \rangle_k$. Rather than the spectrum of the k -independent operator \bar{D}^2 relevant in the path integral, we analysed that of the explicitly k -dependent Laplacian $D^2(\langle g_{\mu\nu} \rangle_k)$; its explicit k -dependence is due to the scale dependence of the on-shell metric. Our argument reveals that the effective spacetime with the running on-shell metric cannot support harmonic modes of arbitrarily fine angular resolution.

This phenomenon is a purely dynamical one; the finite resolution is not built in at the kinematical (i.e. $\gamma_{\mu\nu}$ -) level, as would be the case, for instance, if the fundamental theory was defined on a lattice. It is also important to stress that, if the non-Gaussian fixed point exists, Green's functions $G_n(x_1, x_2, \dots, x_n)$ can be made well defined at all non-coincident points, i.e. for arbitrarily small coordinate distances among x_i^μ 's. Those Green's functions contain information even about angular scales smaller than $\Delta\phi_{\min}$, in particular they 'know' about the asymptotic safety of the theory which manifests itself only at scales $k \gg k_T$.

In fact, the argument leading to the finite resolution $\Delta\phi_{\min}$ is fairly independent of the high energy behaviour of the theory. The crucial ingredient in the above reasoning was the occurrence of a minimum value for $\lambda(k)$. This minimum occurs as a direct consequence of the k^4 -running of $\Lambda(k)$ given in equation (2). However, this k^4 -running occurs already in standard perturbation theory, simply reflecting the quartic divergences of all vacuum diagrams. From this point of view, our argument is rather robust.

The upper bound on the angular momentum-like quantum number n is reminiscent of the 'fuzzy sphere' constructed in [35]. While in the case of the fuzzy sphere the finite angular resolution is put in 'by hand', in the present case it emerges as a consequence of the quantum gravitational dynamics.

It is instructive to ask which proper length would be ascribed to $\Delta\phi_{\min}$ by an observer using the macroscopic, classical metric $\langle g_{\mu\nu} \rangle_{k_{\text{lab}}}$, where k_{lab} is any scale in the classical regime in which G and Λ do not run (in figure 1 between the points P_1 and P_2). We denote this proper length by L_{\min}^{macro} ; it obeys $L_{\min}^{\text{macro}} = r(k_{\text{lab}})\Delta\phi_{\min}$. Using equations (9), (15) and (2), and assuming $k_{\text{lab}} \ll k_T$, we obtain

$$L_{\min}^{\text{macro}} = \frac{\pi}{k_T} \sqrt{\frac{\Lambda(k_T)}{\Lambda(k_{\text{lab}})}} = \frac{\pi}{k_T} \left[\frac{2}{1 + (k_{\text{lab}}/k_T)^4} \right]^{1/2} \approx \sqrt{2\pi} k_T^{-1}. \quad (16)$$

Remarkably, this minimal proper length is different in general from the Planck length which is usually thought to set the minimal length scale. In fact, L_{\min}^{macro} can be much larger than $\ell_{\text{Pl}} \equiv m_{\text{Pl}}^{-1}$. The trajectory realized in nature seems to be an extreme example: it has $k_T^{-1} \approx 10^{30} \ell_{\text{Pl}} \approx 10^{-3}$ cm, and L_{\min}^{macro} is of the same order of magnitude. Should we therefore expect to find an L_{\min}^{macro} of the order of 10^{-3} cm in the real world? The answer is no, most probably. See point (ii) in the discussion at the end of the paper.

4. Intrinsic distance and scale doubling

In fractal geometry and any framework involving a length scale-dependent metric, one can try to define an 'intrinsic' distance of any two points x and y by adjusting the resolving power of the 'microscope' in such a way that the length scale it resolves equals approximately the, yet to be determined, intrinsic (geodesic) distance from x to y .¹ To be concrete, let us fix two points x and y and let us try to assign to them a cutoff scale $k \equiv k(x, y)$ which satisfies

¹ This kind of dynamical adjustment of the resolution has also been used in the RG improvement of black hole [36] and cosmological [37–42] spacetimes; see in particular [41].

$$L_{k(x,y)}(x, y) = \ell(k(x, y)). \tag{17}$$

Equation (17) is a self-consistency condition for $k(x, y)$: the LHS of (17) is the distance from x to y as seen by a microscope with $k = k(x, y)$, and the RHS is precisely the resolution of this microscope. If (17) has a unique solution $k(x, y)$, one defines the intrinsic distance by setting $L_{\text{in}}(x, y) \equiv L_{k(x,y)}(x, y)$. Since $\ell(k) = \pi/k$, this distance is essentially the inverse cutoff scale: $L_{\text{in}}(x, y) = \pi/k(x, y)$.

Let us evaluate the self-consistency condition (17). Without loss of generality, we may again assume that x and y are located on the equator of S^4 so that equation (10) applies. Then, by virtue of (13), equation (17) boils down to the following implicit equation for $k(x, y)$:

$$\lambda(k(x, y)) = \frac{3}{\pi^2} |\phi(x) - \phi(y)|^2. \tag{18}$$

Recalling the properties of the function $\lambda(k)$ for a Type IIIa trajectory, we see that (18) does *not* admit a unique solution for $k(x, y)$. If x and y are such that $|\phi(x) - \phi(y)| < \Delta\phi_{\text{min}}$ it possesses no solution at all, and if $|\phi(x) - \phi(y)| > \Delta\phi_{\text{min}}$ it has at least two solutions. Staying away from the deep UV and IR regimes, every solution $k(x, y) < k_T$ on the lower branch of the RG trajectory has a partner solution $k(x, y)^\sharp > k_T$ on its upper branch. As a result, the intrinsic distance of x and y is either undefined, or there exist at least two different lengths which satisfy the self-consistency condition (17).

In the linear regime where $k^\sharp = k_T^2/k$, the two lengths $L_{\text{in}}(x, y) = \pi/k(x, y)$ and $L_{\text{in}}(x, y)^\sharp = \pi/k(x, y)^\sharp$ are related by

$$L_{\text{in}}(x, y)^\sharp = \frac{L_T^2}{L_{\text{in}}(x, y)}, \tag{19}$$

where $L_T \equiv \pi/k_T$. If $L_{\text{in}}(x, y)$ is large compared to the turning point length scale L_T , the ‘dual’ scale $L_{\text{in}}(x, y)^\sharp$ is small. In the extreme case, when applied to nature’s RG trajectory, the duality (19) would even exchange the Planck regime with the Hubble regime: $L_{\text{in}}(x, y) \approx H_0^{-1}$ implies $L_{\text{in}}(x, y)^\sharp \approx l_{\text{Pl}}$.

This ‘doubling’ of k -scales, again, is due to the ‘back bending’ of the RG trajectory at the turning point T which implies that the function $\lambda = \lambda(k)$ assumes a minimum at a finite scale $k = k_T$. Only the trajectories of Type IIIa possess a turning point of this kind, and this is one of the reasons why they are particularly interesting and we restricted our discussion to them.

5. Discussion

While its origin is quite clear, the physical implications of the scale doubling and the duality symmetry are somewhat mysterious. To some extent, the difficulty of giving a precise physical meaning to them is related to the fact that one actually should define the ‘resolution of the microscope’ in terms of realistic experiments rather than the perhaps too strongly idealized mathematical model of a measurement based upon the COMs. For various reasons, it seems premature to assign direct observational relevance to the minimal angular resolution and the scale doubling.

- (i) Only the *coordinate* distance $\Delta\phi(k)$ assumes a minimum, but not the corresponding *proper* distance computed with the running metric $\langle g_{\mu\nu} \rangle_k$. In particular, the resolution function $\ell(k) = \pi/k$ is exactly the same as in flat space. Nevertheless, the COM microscope is unable to distinguish points with an angular separation below $\Delta\phi_{\text{min}}$!
- (ii) Our analysis applies to pure gravity. In presence of matter, the ‘fuzziness’ of S^4 can become visible probably only at scales where the cosmological constant dominates the energy density. In particular, the fuzziness might be masked by the backreaction of a realistic measuring apparatus on the spacetime structure.

- (iii) As for a possible physical significance of the duality symmetry, it is to be noted that the two scales which it relates, $k < k_T$ and $k^\sharp > k_T$, have a rather different status as far as quantum fluctuations about the mean field metric $\langle g_{\mu\nu} \rangle_k$ are concerned. The structure of the exact RG equation is such that the larger the fluctuations, the stronger the renormalization effects are. As a result, the metric fluctuations about $\langle g_{\mu\nu} \rangle_{k^\sharp}$ on the upper branch are certainly larger than at the dual point on the lower branch of the RG trajectory.

Clearly, more work is needed in order to understand these rather intriguing issues better. We hope to return to them elsewhere.

References

- [1] For a review, see Garay L J 1995 *Int. J. Mod. Phys. A* **10** 145 and references therein
- [2] Weinberg S 1979 *General Relativity, An Einstein Centenary Survey* ed S W Hawking and W Israel (Cambridge: Cambridge University Press)
- Weinberg S 1997 *Preprint* [hep-th/9702027](#)
- [3] Reuter M 1998 *Phys. Rev. D* **57** 971 (*Preprint* [hep-th/9605030](#))
- [4] Dou D and Percacci R 1998 *Class. Quantum Grav.* **15** 3449
- [5] Lauscher O and Reuter M 2002 *Phys. Rev. D* **65** 025013 (*Preprint* [hep-th/0108040](#))
- [6] Reuter M and Saueressig F 2002 *Phys. Rev. D* **65** 065016 (*Preprint* [hep-th/0110054](#))
- [7] Lauscher O and Reuter M 2002 *Phys. Rev. D* **66** 025026 (*Preprint* [hep-th/0205062](#))
- [8] Lauscher O and Reuter M 2002 *Class. Quantum Grav.* **19** 483 (*Preprint* [hep-th/0110021](#))
- [9] Lauscher O and Reuter M 2002 *Int. J. Mod. Phys. A* **17** 993 (*Preprint* [hep-th/0112089](#))
- [10] Souma W 1999 *Prog. Theor. Phys.* **102** 181
- [11] Percacci R and Perini D 2003 *Phys. Rev. D* **67** 081503
- Percacci R and Perini D 2003 *Phys. Rev. D* **68** 044018
- Codello A and Percacci R 2006 *Preprint* [hep-th/0607128](#)
- [12] Perini D 2004 *Nucl. Phys. Proc. Suppl. C* **127** 185
- [13] Reuter M and Saueressig F 2002 *Phys. Rev. D* **66** 125001 (*Preprint* [hep-th/0206145](#))
- Reuter M and Saueressig F 2004 *Fortschr. Phys.* **52** 650 (*Preprint* [hep-th/0311056](#))
- [14] Litim D 2004 *Phys. Rev. Lett.* **92** 201301
- Fischer P and Litim D 2006 *Preprint* [hep-th/0602203](#)
- [15] Bonanno A and Reuter M 2005 *J. High Energy Phys.* JHEP02(2005)035 (*Preprint* [hep-th/0410191](#))
- [16] Percacci R and Perini D 2004 *Class. Quantum Grav.* **21** 5035 (*Preprint* [hep-th/0401071](#))
- [17] Percacci R 2004 *Preprint* [hep-th/0409199](#)
- [18] Forgács P and Niedermaier M 2002 *Preprint* [hep-th/0207028](#)
- Niedermaier M 2002 *J. High Energy Phys.* JHEP12(2002)066
- Niedermaier M 2003 *Nucl. Phys. B* **673** 131
- [19] Wilson K G 1971 *Phys. Rev. B* **4** 3174
- Wilson K G 1971 *Phys. Rev. B* **4** 3184
- Wilson K G 1973 *Phys. Rev. D* **7** 2911
- [20] Niedermaier M and Reuter M to appear in Living Reviews of Relativity
- [21] Wetterich C 1993 *Phys. Lett. B* **301** 90
- [22] Reuter M and Wetterich C 1994 *Nucl. Phys. B* **417** 181
- Reuter M and Wetterich C 1994 *Nucl. Phys. B* **427** 291
- Reuter M and Wetterich C 1993 *Nucl. Phys. B* **391** 147
- Reuter M and Wetterich C 1993 *Nucl. Phys. B* **408** 91
- Reuter M 1996 *Phys. Rev. D* **53** 4430
- Reuter M 1997 *Mod. Phys. Lett. A* **12** 2777
- [23] For a review, see Berges J, Tetradis N and Wetterich C 2002 *Phys. Rep.* **363** 223
- Wetterich C 2001 *Int. J. Mod. Phys. A* **16** 1951
- [24] Reuter M and Schwindt J 2006 *J. High Energy Phys.* JHEP01(2006)070 (*Preprint* [hep-th/0511021](#))
- [25] Reuter M and Weyer H 2004 *J. Cosmol. Astropart. Phys.* JCAP12(2004)001 (*Preprint* [hep-th/0410119](#))
- [26] Litim D F and Pawłowski J M 1998 *Phys. Lett. B* **435** 181
- [27] Falkenberg S and Odintsov S 1998 *Int. J. Mod. Phys. A* **13** 607
- [28] Lauscher O and Reuter M 2005 *J. High Energy Phys.* JHEP10(2005)050 (*Preprint* [hep-th/0508202](#))

- [29] ben-Avraham D and Havlin S 2004 *Diffusion and Reactions in Fractals and Disordered Systems* (Cambridge: Cambridge University Press)
- [30] Ambjørn J, Jurkiewicz J and Loll R 2004 *Phys. Rev. Lett.* **93** 131301
- [31] Ambjørn J, Jurkiewicz J and Loll R 2005 *Phys. Lett. B* **607** 205
- [32] Ambjørn J, Jurkiewicz J and Loll R 2005 *Phys. Rev. Lett.* **95** 171301
Ambjørn J, Jurkiewicz J and Loll R 2005 *Phys. Rev. D* **72** 064014
Ambjørn J, Jurkiewicz J and Loll R 2006 *Contemp. Phys.* **47** 103
- [33] Kawai H and Ninomiya M 1990 *Nucl. Phys. B* **336** 115
Floresani R and Percacci R 1995 *Nucl. Phys. B* **436** 141
Antoniadis I, Mazur P O and Mottola E 1998 *Phys. Lett. B* **444** 284
- [34] Mandelbrot B 1983 *The Fractal Geometry of Nature* (New York: Freeman)
- [35] Madore J 1992 *Class. Quantum Grav.* **9** 69
- [36] Bonanno A and Reuter M 2000 *Phys. Rev. D* **62** 043008 (Preprint [hep-th/0002196](#))
Bonanno A and Reuter M 2006 *Phys. Rev. D* **73** 083005 (Preprint [hep-th/0602159](#))
Bonanno A and Reuter M 1999 *Phys. Rev. D* **60** 084011 (Preprint [gr-qc/9811026](#))
- [37] Bonanno A and Reuter M 2002 *Phys. Rev. D* **65** 043508 (Preprint [hep-th/0106133](#))
Reuter M and Saueressig F 2005 *J. Cosmol. Astropart. Phys.* JCAP09(2005)012 (Preprint [hep-th/0507167](#))
- [38] Bonanno A and Reuter M 2002 *Phys. Lett. B* **527** 9 (Preprint [astro-ph/0106468](#))
Bonanno A and Reuter M 2004 *Int. J. Mod. Phys. D* **13** 107 (Preprint [astro-ph/0210472](#))
Bentivegna E, Bonanno A and Reuter M 2003 *J. Cosmol. Astropart. Phys.* JCAP01(2004)001 (Preprint [astro-ph/0303150](#))
- [39] Bonanno A, Esposito G and Rubano C 2003 *Gen. Rel. Grav.* **35** 1899
Bonanno A, Esposito G and Rubano C 2004 *Class. Quantum Grav.* **21** 5005
Bonanno A, Esposito G, Rubano C and Scudellaro P 2006 *Class. Quantum Grav.* **23** 3013
- [40] Reuter M and Weyer H 2004 *Phys. Rev. D* **69** 104022 (Preprint [hep-th/0311196](#))
- [41] Reuter M and Weyer H 2004 *Phys. Rev. D* **70** 124028 (Preprint [hep-th/0410117](#))
- [42] Moffat J 2005 *J. Cosmol. Astropart. Phys.* JCAP05(2005)003 (Preprint [astro-ph/0412195](#))
Brownstein J R and Moffat J 2006 *Astrophys. J.* **636** 721
Brownstein J R and Moffat J 2006 *Mon. Not. R. Astron. Soc.* **367** 527